Confidence Interval for a Mean using the *t* Distribution

These notes present the method for finding a confidence interval estimate of a population mean when the population standard deviation is not known.

The sample mean, $\bar{x}$ is the best point estimate of the population mean. If we knew σ, we could use a Normal distribution to find the confidence interval. However, σ is generally unknown. If this is the case, and the following conditions are met

1. The sample is a simple random sample.
2. Either the sample is from a normally distributed population or n > 30

then we can use what is called the *Student t* distribution. Because we do not know the value of σ, we estimate it with the value of the sample standard deviation s.

Student t Distribution

If a population has a normal distribution, then the distribution of

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

is a Student t distribution for all samples of size n. A Student t distribution, often referred to as a t distribution, is used to find critical values denoted by t_c.

Important Properties of the Student t Distribution

- The Student t distribution has the same general symmetric bell shape as the standard normal distribution, but it reflects the greater variability (with wider distributions) that is expected with small samples.
- The Student t distribution is different for different sample sizes. It is a family of bell shaped curves determined by a parameter called the degrees of freedom (df).
  df = n – 1
- The total area under the t-curve is 1 or 100%
- The Student t distribution has a mean of t = 0 (just as the standard normal distribution has a mean of z = 0). The standard deviation of the Student t distribution varies with the sample size, but it is greater than 1 (unlike the standard normal distribution, which has $\sigma = 1$).
- As the sample size n gets larger, the Student t distribution gets closer to the standard normal distribution. When n > 30, values of the t distribution are almost the same as those of the Standard Normal.

Degrees of Freedom

The number of *degrees of freedom* for a collection of sample data is the number of sample values that can vary after certain restrictions have been imposed on all data values.

degrees of freedom (df) = n – 1

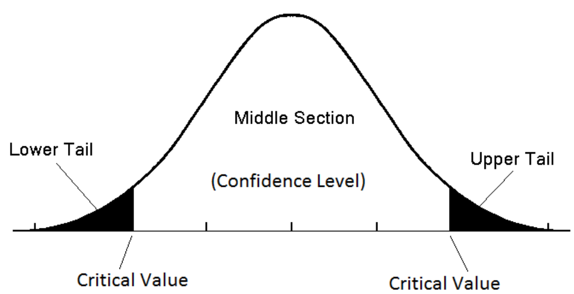
If I know the mean of 3 numbers is equal to 10, do I know what those numbers are? No. The first of the three numbers can be anything (it has complete freedom). Once the first number is determined, the second can still be anything (it has complete freedom). But if I know the first and second value (say 3 and 46), and that the mean has to be 10:

$$\frac{3+46+?}{3} = 10$$

The only number that will work is -19 (the third number has no freedom). So n – 1 = 3 – 1 = 2 values are free, and there are 2 degrees of freedom.

Finding a Critical Value

The confidence level is incorporated into the confidence interval through a value known as the *critical value*. The confidence level is the area under the distribution curve that is between the critical values.



To find a critical value from a *t* distribution, you need to know the confidence level and the degrees of freedom. In our table, the confidence level is the probability in the "central area". Find the correct confidence level and read down the table to the appropriate degrees of freedom.

<u>Example</u>: A sample of size n = 23 is a simple random sample selected from a normally distributed population. Find the critical value t_c corresponding to a 95% confidence level.

> Because n = 23, the number of degrees of freedom is given by n – 1 = 22.
> The "central area" is 95%, read down to 22 degrees of freedom.
> t_c = 2.074

Margin of Error

Now that we know how to find critical values denoted by t_c we can describe the margin of error E and the confidence interval.

Recall that the margin of error, E, is found by multiplying the critical value and the standard error.

$$E = t_c \frac{s}{\sqrt{n}}$$

Both critical values (one positive, one negative) are not necessary for the formula. We only need to use the positive value.

Procedure for Constructing a Confidence Interval for μ (With σ Unknown)

1. Verify that the requirements are satisfied. (We have a simple random sample, and either the population appears to be normally distributed or n > 30.)
2. Using n – 1 degrees of freedom, find the critical value t_c that corresponds to the desired confidence level.
3. Evaluate the margin of error

$$E = t_c \frac{s}{\sqrt{n}}$$

4. Using the value of the calculated margin of error E and the value of the sample mean $\bar{x}$, find the values of $\bar{x}$ – E and $\bar{x}$ + E. Substitute those values in the general format for the confidence interval:

$\bar{x} - E < \mu < \bar{x} + E$

or $\bar{x} \pm E$

or $(\bar{x} - E, \bar{x} + E)$

Round the resulting confidence interval limits. If using the original set of data, round to one more decimal place than is used for the original set of data. If using summary statistics (n, $\bar{x}$, s), round the confidence interval limits to the same number of decimal places used for the sample mean.

5. Interpret the Confidence Interval (see below).

Interpreting a Confidence Interval

We must be careful to interpret confidence intervals correctly. There is a correct interpretation and many different and creative wrong interpretations of the confidence interval. We must also distinguish between interpretations of the confidence *interval* and the confidence *level*.

Confidence level, C: If we repeatedly take samples of the same size, construct a C% confidence interval from each, then C% of those confidence intervals will contain the true value of the parameter.

Confidence Interval: With C% confidence, we can say that the true (parameter you're estimating) for (population description) is between (lower limit, $\bar{x}$ – E) (units) and (upper limit, $\bar{x}$ + E) (units).

*Lots of blanks to fill in, but with practice it becomes very easy.

Here's an example interpretation:

We can say with 90% confidence that the true mean for the forearm length of all US males is between 24.33 cm and 26.67 cm.

What NOT to say:

- Do not say 95% of the data are in this interval.
- Do not say there is a 95% chance that the true mean is in our interval.
- Do not say with 95% probability the true mean is in our interval.
- Do not say there is a 95% chance another sample's mean would fall in the interval.

Example

A study was done to determine the average number of homes that a homeowner owns in his or her lifetime. For the 65 random homeowners surveyed, the sample mean was 4.2 and the sample standard deviation was 2.1. Calculate the 95% confidence interval for the true mean number of homes that a person owns in his or her lifetime.

Solution:

Here the level of confidence = 1- α = 0.95 → α = 0.05

$\bar{x} = 4.2, \quad s = 2.1 \text{ and } n = 65 => df = n - 1 = 64$ (round down on table to 60)

$$t_{\frac{\alpha}{2},df} = t_{0.025,60} = 2.00$$

$$\widehat{SE} = \frac{s}{\sqrt{n}} = \frac{2.1}{\sqrt{65}} = 0.2605$$

$$E = (2.000) * (0.2605) = 0.5210$$

$$\bar{x} - E \leq \mu_X \leq \bar{x} + E \quad \rightarrow \quad (3.679, 4.721)$$

We are 95% confident that the true mean number of homes that a homeowner has in their lifetime is between 3.679 and 4.721.